Borel Perfect Fractional Matchings

In Quasitransitive Amenable Graphs

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Descriptive Combinatorics

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- Baire measurable w/ respect to a compatible Polish topology τ on V

Familiar Graph Problems

A **perfect matching** of a graph $G = (V, E)$ is a function $M : E \to \{0, 1\}$ such that for every vertex v:

$$\sum_{e \in N_E(v)} M(e) = 1$$

Linear Relaxations of Graph Problems

Integer Programming $\longrightarrow$ Linear Programming:

- Perfect Matching $\rightarrow$ Perfect Fractional Matching

Relaxed Graph Problems

A **perfect** **fractional** **matching** of a graph $G = (V, E)$ is a function $m : E \to [0, 1]$ such that for every vertex v:

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Matching Polytope

For finite graphs, the set of perfect fractional matchings forms a polytope in $\mathbb{R}^E$ called the **matching polytope**.

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For finite bipartite graphs, the extreme points of the matching polytope are exactly the set of perfect matchings.

Matching Polytope

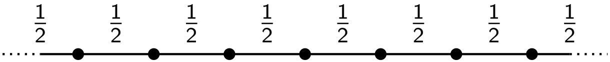
In particular, the existence of a perfect fractional matching on a bipartite graph implies the existence of an integral perfect matching.

Descriptive Fractional Graph Theory

Q: Does something similar hold in the Descriptive context?

Descriptive Fractional Graph Theory

A: Not in general (Laczkovich):



Measurable Fractional Matchings

Theorem (M.Sabok, M. Bowen, G. Kun)

If a G is a hyperfinite one-ended bipartite graphing which admits an a.e positive measurable perfect fractional matching m, then G admits a measurable perfect matching.

Existance of Perfect Fractional Matchings

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Existance of Perfect Fractional Matchings

Q: Say a Borel graph admits a classical (non-descriptive) perfect fractional matching. Does that graph admit a descriptive perfect fractional matching?
A:

- Yes in the measure context for hyperfinite graphings. (M. Sabok, T. Cieśla).
- No in general for the Borel and Baire measurable context. (A. Bernshteyn, F. Weilacher, 2024+)

The Quasitransitive Amenable Context

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If G is a locally finite Borel graph with quasi-transitive amenable components then if G admits a perfect fractional matching, it admits a Borel perfect fractional matching.

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Theorem (M.)

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- Quasitransitive: On each component of G, there are only finitely many $\mathrm{Aut}(G)$ orbits.
- Amenable: If for any component C of G,

$$\inf\{\frac{|\partial_V(S)|}{|S|} : S \subseteq C \text{ finite connected subset}\} = 0$$

Proof Sketch

- Construct **Half-Edge Weighted Borel Multigraph** $M = (V, E, s, w)$.

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- Show that $\text{Im}(\varphi) \subseteq \text{dom}(m')$
- Show $m = m' \circ \varphi$ is a perfect fractional matching on G.

Proof Sketch

A **Borel Multigraph** $M = (V(M), E(M), s)$ is a pair of standard Borel spaces $V(M)$ and $E(M)$ along with an Borel endpoint map $s : E \to [V]^2 \cup [V]^1$.

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A half-edge weighting is a map $w : V(M) \times E(M) \to \mathbb{N}$, where $w(v, e) > 0$ only when v is incident to e.

Constructing M

- Let $V(M)$ be sequences of finite rooted graphs $(F_0, F_1, \ldots, r)$ such that $F_n = F_{n+1} \upharpoonright B_n(r)$.

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- Say $(F_0, \ldots, r) \in s((H_0, \ldots, e))$ if for all n, there exists a vertex r_n in H_n incident to e such that $F_n = H_n \upharpoonright B_n(r_n)$

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- Define $w((F_0, \dots, r), (H_0, \dots, e))$ to be the minimum k such that for all n there are k edges e in F_{n+1} incident to r with $H_n = F_{n+1} \restriction B_n(e)$.

Finding Homomorphisms

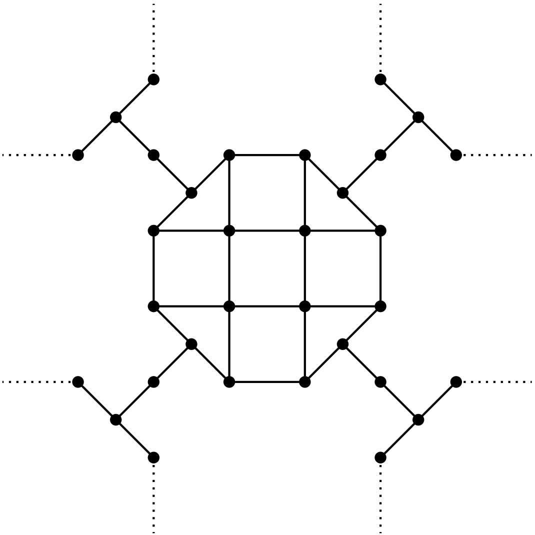
Let G be a locally finite Borel graph, define homomorphism (φ_V, φ_E) via:

$$\varphi_V(x) = (G \upharpoonright B_0(x), G \upharpoonright B_1(x) \ldots, x)$$
$$\varphi_E(g) = (G \upharpoonright B_0(g), G \upharpoonright B_1(g) \ldots, g)$$

(φ_V, φ_E) sends components of G to components of M. Moreover, for any $e \in E(M)$:

$$w(\varphi_V(x), e) = |g \in N_E(v) : \varphi_E(g) = e|$$

Example:



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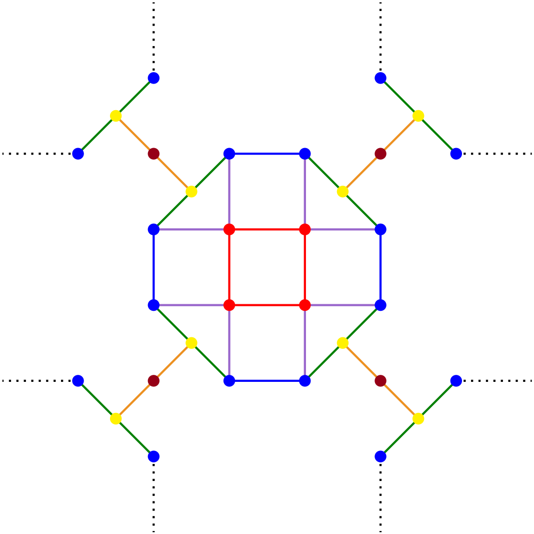

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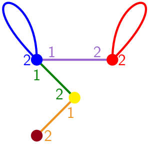
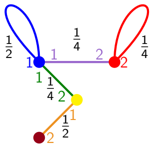


Weighted fractional matchings

A Borel **weighted perfect fractional matching** on a half-edge weighted Borel multigraph is a Borel function $m' : E \to [0, 1]$ such that:

$$\sum_{e \in N_E(v)} m'(e)w(v, e) = 1$$

Example:



Perfect Fractional matching on M

Fix a Borel choice of weighted perfect fractional matching on all finite components of M that admit one. Call this perfect fractional matching m'.

Automorphism invariant perfect fractional matchings

Theorem (M. Salvatori 92)

If H is a locally finite countable graph that is quasitransitive and amenable, then $\mathrm{Aut}(H)$ is amenable.

- Integrate classical perfect fractional matching w/ mean on $\mathrm{Aut}(H)$
- This gives well-defined weighted perfect fractional matching on $\mathrm{Im}(\varphi_E)$.

Last Step

Show $m = m' \circ \varphi_E$ is f.p.m

$$\begin{aligned}
\sum_{g \in N_E(x)} m(g) &= \sum_{g \in N_E(x)} m' \circ \varphi_E(g) \\
&= \sum_{e \in N_E(\varphi_V(x))} m'(e)|g \in N_E(x) : \varphi_E(g) = e| \\
&= \sum_{e \in N_E(\varphi_V(x))} m'(e)w(e) \\
&= 1
\end{aligned}$$

Thanks!